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A Numerical Method for the Evaluation of an Equilibrium Configuration of a Toroidal Pinch

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The equilibrium condition for a toroidal pinch is formulated as a nonlinear operator equation for the parameter functions representing the plasma boundary. This operator equation is transformed into an optimization problem, and a finite-dimensional approximation to the optimization problem is given which is treated by an algorithm proposed by Brent ("Algorithms for Minimization without using Derivatives," Prentice-Hall, Englewood Cliffs, N.J., 1973). This procedure turns out to be numerically stable and reasonably fast. This approach may easily be generalized to treat a variety of free boundary problems in two and three dimensions.

1. INTRODUCTION

Equilibrium configurations of plasma containment devices have received considerable attention during the last two decades, under computational [1, 3, 6–9, 13, 20, 21] as well as theoretical [16, 18, 19] aspects. While the older computations assume the plasma pressure to be continuous, the actual pressure gradients are so large [8, 12] that it seems natural to introduce surfaces of discontinuous pressure into the mathematical model. The problem then consists of computing the exact position of the discontinuity. This problem has been treated in [3, 9, 19]. We now develop a new numerical method for solving this problem for a hydrodynamical model of the toroidal pinch with surface current. To keep computing time down we restrict the computations to the case of rotational symmetry. We then obtain the equilibrium condition as the solution of a nonlinear optimization problem which is solved iteratively.

In contrast to [3, 9, 20] we use integral equation methods to solve the differential equation part of the problem. The respective merits of finite elements and integral equations have frequently been discussed, in the present situation the latter have two definite advantages: They require less storage capacity, and in the case of rotational symmetry with analytic boundaries we easily get very high order discretization formulas and therefore rapid convergence with an increasing number of points.

For the parameters representing the free boundary we choose the coefficients of a cosine series of a radius length (see Fig. 2) thus ensuring analyticity of every trial

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boundary. This method seems superior to a representation by a certain number of points on the boundary.

The optimization procedure we used converges somewhat faster than the steepest descent method in [3], which is comparable with the procedure used in [20]. It could probably be incorporated in the method of [3] without too much effort. One drawback of our procedure relative to the one in [3] is that it is unable to distinguish between stable and unstable configurations, so stability has to be tested separately.

2. The Toroidal Pinch

The setup of the toroidal pinch is the following (for a more detailed description see [10]): A plasma P with surface S_1 is separated from a toroidal ideal conductor S_2 by a vacuum region V. Currents flowing on S_1 and S_2 induce magnetic fields B in P and B' in V such that the gas pressure p of the plasma is balanced by the magnetic pressure of the field B'. For our computations, the configuration is assumed to have rotational symmetry with respect to the y-axis, so its geometry is uniquely defined by the intersections C_1 and C_2 of S_1 and S_2 with the half plane $\{(x, y, 0) \mid x > 0\} \subset \mathbb{R}^3$. We further assume the curves C_i, $i = 1, 2$, to have a representation by parameter functions

$$\begin{aligned} x_i(\phi) &= M_i + r_i(\phi) \cdot \cos \phi, \\ y_i(\phi) &= r_i(\phi) \cdot \sin \phi, \qquad i = 1, 2, \end{aligned} \tag{2.1}$$

with $M_i \in \mathbb{R}^+$, $r_i \in C^2_{2\pi}$, $r_i(\phi) > 0$ $\forall \phi \in [0, 2\pi]$, where M_2 and r_2 are given while M_1 and r_1 have to be determined (see Figs. 1, 2).

The magnetic fields are determined by the following equations [2, 15]:

$$\operatorname{rot} B = 0 \quad \text{in } P, \tag{2.2}$$
$$\operatorname{div} B = 0 \quad \text{in } P, \tag{2.3}$$
$$\operatorname{grad} p = 0 \quad \text{in } P, \tag{2.4}$$
$$(n_1, B) = 0 \quad \text{in } S_1, \tag{2.5}$$
$$(n_1, B') = 0 \quad \text{in } S_1, \tag{2.6}$$
$$\operatorname{rot} B' = 0 \quad \text{in } V, \tag{2.7}$$
$$\operatorname{div} B' = 0 \quad \text{in } V, \tag{2.8}$$
$$(n_2, B') = 0 \quad \text{in } S_2, \tag{2.9}$$

where n_1 is the normal to S_1 and n_2 to S_2.

From [4, 5] we have the following characterization of the fields B and B': Let $d(x, y, z) := \{x^2 + z^2\}^{1/2}$ and e be the azimuthal unit vector:

$$e(x, y, z) = (-z, 0, x)/d; \tag{2.10}$$

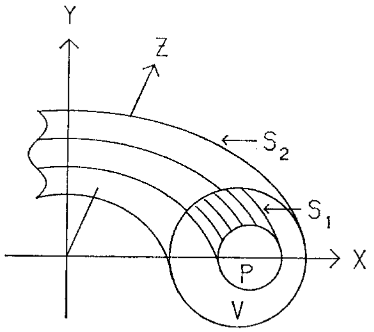

FIG. 1. Geometry of the toroidal pinch. V: Vacuum region; P: plasma region; S_1 : plasma boundary; S_2 : outer conductor.

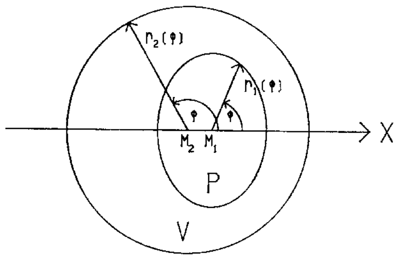

FIG. 2. Cross section of a toroidal pinch.

then

$$B = t/(2\pi d) \cdot e \qquad \text{in } P \tag{2.11}$$

and

$$B' = t'/(2\pi d) \cdot e + \bar{B} \qquad \text{in } V, \tag{2.12}$$

where $t, t' \in \mathbb{R}$ and $\bar{B}$ is a purely meridional field in V with vanishing normal component on S_1 and S_2.

$\bar{B}$ is uniquely determined by a flow condition: Let I be the total azimuthal current in the plasma surface S_1 ; c, the vacuum speed of light; and ds_i , the line element on C_i , $i = 1, 2$. Then we have

$$(c/4\pi) \int_{C_i} (\bar{B}, ds_i) = I, \qquad i = 1, 2. \tag{2.13}$$

The configuration is in an equilibrium state if the magnetic field just balances the gas pressure:

$$B^2 - B'^2 + 8\pi p = 0 \qquad \text{on } S_1 \tag{2.14}$$

or, using (2.11), (2.12)

$$|\bar{B}| = \{8\pi p - (t'^2 - t^2)/(2\pi d)^2\}^{1/2} \qquad \text{on } S_1 . \tag{2.15}$$

Our problem now is the following: Let p, t, t', I, and the shape of S_2 be given. Find an admissible shape of S_1 such that the field $\bar{B}$ defined in V as above fulfills the equilibrium condition (2.15).

Admissible are such shapes that

(1) S_1 is entirely interior to S_2, and

(2) The curve C_1 has a representation as given in (2.1). As $\bar{B}$ and d depend on the shape of S_1, (2.15) thus gives an operator equation for the determination of M_1 and r_1. The fact that M_1 and r_1 are not uniquely determined by the shape of S_1 is of minor importance here; we may eliminate this multiplicity later on by fixing M_1 to a suited value.

For the numerical treatment we reformulate the problem as follows: Let

$$F(M_1, r_1) := \| \, |\bar{B}| - \{8\pi p + (t'^2 - t^2)/(2\pi d)\}^{1/2} \|_2 ; \tag{2.16}$$

then we are looking for a pair (M'_1, r'_1) minimizing F under all admissible pairs (M_1, r_1). Clearly, if $F(M'_1, r'_1) = 0$, we have a solution to the equilibrium problem; if not, such a solution does not exist.

3. The Evaluation of $\bar{B}$

For the evaluation of the equilibrium condition (2.15) we need only know the values $\bar{B}$ takes along the curve C_1. To get these, we use a numerical procedure developed in [4] which can be shown to converge with the methods of [22]. Let

$$w_i(\phi) = |\bar{B}| \cdot \{\dot{x}_i^2(\phi) + \dot{y}_i^2(\phi)\}^{1/2} \qquad \text{on } C_i , \tag{3.1}$$

where $\cdot$ denotes the derivative with respect to ϕ. Then the functions w_i are the unique solutions of the system of integral equations

$$2\pi w_i(\phi) = \sum_{j=1}^{2} \int_0^{2\pi} K_{ij}(\phi, \phi') \cdot w_j(\phi') \cdot d\phi' \tag{3.2}$$

together with the uniqueness condition

$$\int_0^{2\pi} w_2(\phi) \cdot d\phi = I, \tag{3.3}$$

and every solution of (3.2) and (3.3) gives a field which can be found by an integrating process.

The kernels K_{ij} are defined by

$$K_{ij}(\phi, \phi') := \int_0^{2\pi} \frac{[x_i(\phi) \cdot \cos(\psi) - x_j(\phi')] \cdot y_i(\phi) - [y_i(\phi) - y_j(\phi')] \cos(\psi) \cdot x_i(\phi)}{\{x_i^2(\phi) - 2x_i(\phi) \cdot x_j(\phi') \cdot \cos(\psi) + x_j^2(\phi') + [y_i(\phi) - y_j(\phi')]^2\}^{3/2}} \times x_j(\phi') \cdot d\psi \tag{3.4}$$

For $i \neq j$ the kernel K_{ij} is continuous, and for $i = j$ it may be split into

$$K_{ii}(\phi, \phi') = K'_{ii}(\phi, \phi') + K''_{ii}(\phi, \phi') \cdot \ln\left(4 \sin^2 \frac{(\phi - \phi')}{2}\right),$$

where K', K'' are continuous.

This system of integral equations can be discretized through a quadrature formula with equally spaced abscissas as proposed in [14], which exhibits rapid convergence if C_1 and C_2 are analytic curves.

4. The Continuity of the Function F

To show the continuity of F we only have to show that w_1 depends continuously on the choice of r_1 in some proper sense. We define

$$\|f\|'' = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty \qquad \forall f \in C^2_{2\pi}.$$

LEMMA 4.1. *Let M_1 be fixed, r'_1 such that $C'_1 \cap C_2 = \varnothing$, where C'_1 is the curve given by M_1 and r'_1; then there are constants $\delta > 0$; $L_1, L_2, L_3 < \infty$ such that for all r_1 with $\| r_1 - r'_1 \|'' < \delta$ we have*

$$\| K_{ij} - K'_{ij} \|_\infty \leqslant L_1 \cdot \| r_1 - r'_1 \|'', \qquad i, j = 1, 2, \qquad i \neq j, \tag{4.1}$$

$$| K_{11}(\phi, \phi') - K'_{11}(\phi, \phi')| \leqslant \| r_1 - r'_1 \|'' \cdot \left\{ L_2 + L_3 \cdot \left| \ln\left(4 \sin^2 \frac{(\phi - \phi')}{2}\right)\right|\right\}, \tag{4.2}$$

$$K_{22} = K_{22}'; \tag{4.3}$$

where K' is the kernel belonging to M_1, r'_1 and K the kernel belonging to M_1, r_1.

We introduce some notation:

Let $v = (v_1, v_2)$, $v_i \in C[0, 2\pi]$, $i = 1, 2$, and v' the same; then we define

$$(v, v') := \sum_{j=1}^{2} \int_0^{2\pi} v_j(\phi) \cdot v_j'(\phi) \cdot d\phi, \tag{4.4}$$

$$\| v \|_2 := (v, v)^{1/2}, \qquad \| v \|_\infty := \max(\| v_1 \|_\infty, \| v_2 \|_\infty). \tag{4.5}$$

LEMMA 4.2. *Let r'_1 be as before, with v'_1 the solution of the integral equation* (3.2) *under the uniqueness condition*

$$\| v'_1 \|_2 = 1;$$

then there exist constants $\delta > 0$ and $H < \infty$, such for all r_1 with $\| r_1 - r'_1 \|'' < \delta$ we have

$$\| v - v' \|_\infty \leqslant H \cdot \| r_1 - r'_1 \|'', \tag{4.6}$$

where v is the solution of (3.2) *to r_1.*

Proof. In [11] solutions of (3.2) are proved to be Hölder continuous with exponent $\frac{1}{4}$, and an examination of that proof shows the Hölder constant to be a continuous function of r_1. Therefore the solutions v are for all r_1 with $\| r_1 - r'_1 \| < \delta$ equally Hölder continuous, and we get a constant $H_1 < \infty$ such that

$$\left| \int_0^{2\pi} v_j(\phi') \cdot \ln \left(4 \cdot \sin^2 \left(\frac{\phi' - \phi}{2}\right)\right) \cdot d\phi' \right| < H_1 , \qquad \phi \in [0, 2\pi),\ j = 1, 2, \tag{4.7}$$

with H_1 not depending on r_1. This together with Lemma 4.1 gives us the existence of a constant $H_2 < \infty$ such that

$$\left\| \frac{1}{2\pi} \sum_{j=1}^{2} \int_0^{2\pi} (K_{ij} - K'_{ij}) \cdot v_j \cdot d\phi' \right\|_2 \leqslant H_2 \cdot \| r_1 - r'_1 \|''. \tag{4.8}$$

Now v and v' are solutions of (3.2) and therefore

$$v_i - v'_i = \frac{1}{2\pi} \sum_{j=1}^{2} \int_0^{2\pi} (K_{ij} - K'_{ij})\, v_j + K'_{ij}(v_j - v'_j)\, d\phi', \qquad i = 1, 2. \tag{4.9}$$

Now let

$$v = \alpha \cdot v' + \beta \cdot \bar{v} \qquad \text{with} \quad (\bar{v}, v') = 0, \qquad \| \bar{v} \|_2 = 1. \tag{4.10}$$

Adding $(1 - \alpha) \cdot v'$ on either side of (4.9) we get

$$\beta \cdot \bar{v} = -\frac{1}{2\pi} \sum_{j=1}^{2} \int_0^{2\pi} \{(K_{ij} - K'_{ij}) \cdot v_j + \beta \cdot K'_{ij} \bar{v}_j\}\, d\phi'. \tag{4.11}$$

Because 1 is a simple eigenvalue of (3.2) [14] and the integral operator is compact, there exists an $\epsilon > 0$ such that for all g with $(g, v') = 0$ we have

$$\left\| g - \frac{1}{2\pi} \sum_{j=1}^{2} \int_0^{2\pi} K_{ij} \cdot g_j \cdot d\phi' \right\|_2 \geqslant \epsilon \cdot \| g \|_2 , \tag{4.12}$$

so we conclude from (4.11)

$$\left\| \frac{1}{2\pi} \sum_{j=1}^{2} \int_0^{2\pi} (K_{ij} - K'_{ij}) \cdot v_j \cdot d\phi' \right\|_2 \geqslant \beta \cdot \epsilon \tag{4.13}$$

$$\Rightarrow \beta \leqslant \frac{1}{2\pi} \cdot H_2 \cdot d \, \| r_1 - r'_1 \| \cdot \epsilon \tag{4.14}$$

and because of $\alpha^2 + \beta^2 = 1$ there exists an H_3 such that

$$\| v' - v \|_2 = (1 - \alpha)^2 + \beta^2 \leqslant H_3 \cdot \| r_1 - r'_1 \|''. \tag{4.15}$$

Now $v - v'$ is a Hölder continuous function with Hölder exponent $\frac{1}{4}$ and a Hölder constant not depending on r_1, so there exists a constant $H_4 < \infty$ depending only on r'_1 and such that

$$\| v - v' \|_\infty \leqslant H_4 \cdot \| v - v' \|_2 \qquad \forall r_1 \colon \| r_1 - r'_1 \|'' < \delta. \tag{4.16}$$

We now can prove our final result:

THEOREM 4.3. *Let $\bar{v}$ be the solution of* (3.2) *under the uniqueness condition* (3.3); *then $\bar{v}$ depends continuously on r_1.*

Proof.

$$\bar{v} = v \cdot I \Big/ \left\{ \int_0^{2\pi} v_2(\phi) \, d\phi \right\}. \tag{4.17}$$

As a solution of a homogeneous Neumann problem is by definition continuous and has vanishing divergence, v_1 and v_2 have no change of sign; thus

$$\left| \int_0^{2\pi} v_i(\phi) \, d\phi \right| = \int_0^{2\pi} | v_i(\phi) | \, d\phi, \qquad i = 1, 2. \tag{4.18}$$

Because of Lemma 4.2 the integral depends continuously on H, it is strictly positive and thus

$$v \Big/ \left\{ \int_0^{2\pi} v_2(\phi) \, d\phi \right\}$$

depends continuously on r_1, which proves the theorem.

5. The Minimizing Problem

For the numerical solution we have to construct a finite-dimensional approximation of the problem.

We set

$$A^n := \left\{ (M_1, r_1) \,\middle|\, r_1 = \sum_0^n a_i \cos i\phi;\ (M_1, r_1) \text{ admissible} \right\}. \tag{5.1}$$

Each element of A^n represents a different curve C_1, but unless n is small, large changes in the coefficients M_1, $a_0, \ldots, a_n$ may result in only small changes in the geometry of the curve. A^n is therefore not a set in which to solve the optimization problem, but it is suited to fix M_1. We choose n'' small, e.g., $n'' = 2$, and solve the optimization problem in $A^{n''}$. Let (M'', r'') be the solution of this optimization problem. We may expect that we may now look for an optimal solution of the form (M'', r_1), $r_1 \in C^2_{2\pi}$.

We now choose $n > n''$, define

$$A^n := \left\{ (M_1'', r_1) \,\middle|\, r_1 = \sum_0^n a_i \cos i\phi;\ (M_1'', r_1) \text{ admissible} \right\}. \tag{5.2}$$

Again each element of A^n represents exactly one curve C_1, but now small changes in the geometry of the curve are caused by only small changes of the parameters $a_0, \ldots, a_n$; and therefore we can hope to solve the optimization problem with respect to A^n regardless of the magnitude of n, and to get convergence as n increases. This convergence may be tested entirely in terms of convergence of the coefficients.

We now have to minimize the function F, which is a measure for the deviation from equilibrium, over A^n, which may be considered a subset of $\mathbb{R}^{n+1}$. Thus we have a nonlinear optimization problem with nonlinear restrictions. Actually we know from physics that for any setup the equilibrium configuration is such that the distance between C_1 and C_2 is strictly positive, the curvature of C_1 is uniformly bounded, and for a proper choice of M_1 the function r_1 is strictly positive, so the solution of the minimizing problem is in the interior of the set of admissible values $a_0, \ldots, a_n$, and we do not have to bother with the restrictions if only we choose our starting point carefully and make sure that any search steps taken are not too large.

The accuracy to which a solution is found depends, naturally, on the accuracy to which the function F is computed. The procedure of [14] gives us values of $|B|$ at $2m$ different points on C_1, of which, because of the symmetry of the setup with respect to x–z-plane, only $m + 1$ are independent.

For a sufficiently smooth setups, e.g., C_2 a circle, C_1 slightly distorted from a circle, $m > 2n$ appeared to warrant a sufficient accuracy of the function F to use the full accuracy of the minimizing procedure, while for more complex geometries a larger choice of m was needed. The choice $m = n$ was tested, too. This means treating directly the discretized operator equation of the equilibrium problem without introducing the optimization concept. This procedure proved to be highly unstable, and proper results could not be obtained.

For the minimization procedure we tested the version of the D–F–P-algorithm given in [17], where the derivatives were replaced by difference quotients, and an algorithm of Brent [5] not using derivatives. The latter proved superior in speed and accuracy. The reason for this seems to be the numerical errors in the calculated derivatives in the D–F–P-algorithm. All final results were therefore calculated with the procedure of [5]. The parameters of the procedure were adjusted as follows: The problem was supposed to be well conditioned, to be possibly badly scaled and the maximum step size was overestimated about twice, which proved to give the fastest convergence.

To improve the scaling of the problem we made the following change: Instead of the a_i, $i = 0,\dots, n$, we used parameters $b_i = a_i \cdot 8^i$, $i = 0,\dots, n$, with the result that the optimal parameters were all of the same magnitude. The value 8 was chosen to avoid rounding errors.

6. The Results

Computations have been done for a variety of different geometries and different values of the parameters t, t', and p, choosing $n = 5$ and $m = 12$ in most cases. All computations have been done on the Univac 1108 of the GWD Göttingen in single

TABLE I

	Number								
	1a	1b	1c	2a	2b	2c	3a	3b	3c
l/c	1.	1.	1.	1.	1.	1.	1.	1.	1.
p	10.	20.	30.	10.	20.	30.	5.	10.	15.
t	0.	0.	0.	0.	0.	0.	1.3333	2.6667	4.
t'	0.	0.	0.	0.	0.	0.	2.	4.	6.
M_2	1.	1.	1.	1.	1.	1.	1.	1.	1.
b_0	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22
b_2	0.0	0.0	0.0	0.03	0.03	0.03	0.0	0.0	0.0
M_1	1.0188	1.0285	1.0362	1.0188	1.0278	1.0340	1.0090	1.0190	1.0258
a_0	0.1262	0.0892	0.0723	0.1262	0.0892	0.0728	0.1853	0.1360	0.1156
a_1	0.0012	0.0006	−0.0002	0.0011	0.0008	0.0003	−0.0007	0.0011	0.0008
a_2	−0.0008	−0.0005	−0.0004	−0.0008	−0.0005	−0.0004	−0.0008	−0.0011	−0.0011
a_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$F/\Vert B\Vert_2$	1.4[a] 10^{-5}	8.[a] 10^{-6}	8.[a] 10^{-6}	4.[b] 10^{-4}	5.[b] 10^{-6}	1.6[b] 10^{-6}	1.2[c] 10^{-4}	5.4[c] 10^{-5}	2.4[c] 10^{-5}

[a] See Fig. 3a.
[b] See Fig. 3b.
[c] See Fig. 3c.

precision (27 bits of the mantissa). In equilibrium, the plasma torus always has an almost circular cross section even if the cross section of the conductor is far from being circular, and its major axis is shifted outward from the major axis of the conductor. This shift is such that the two rings which the vacuum region cuts out of the x–z plane have approximately the same area. Stability tests have been performed with the equilibrium configurations obtained using the method of [15]. They showed the equilibrium configurations to be slightly more stable than neighboring configurations, but the difference is not very significant. For some results see Table I and Fig. 3.

a

b

c

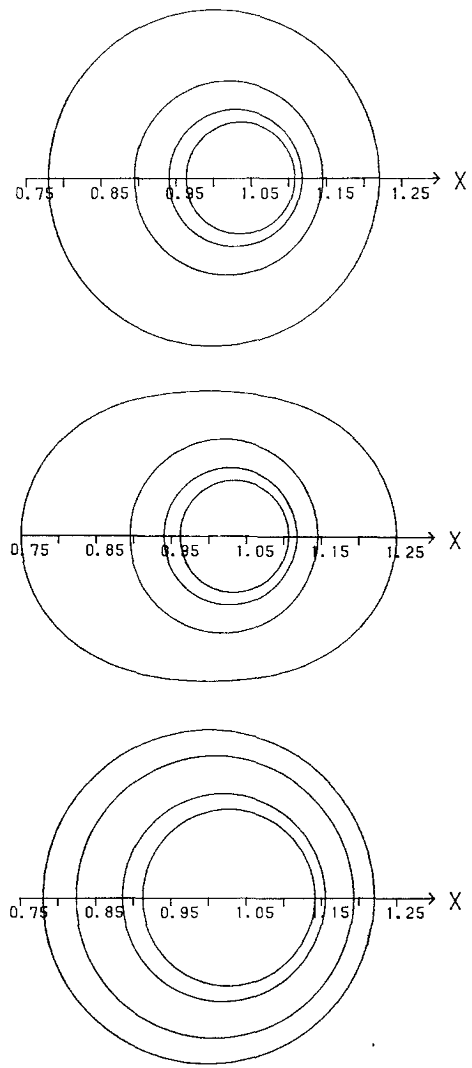

FIG. 3. Conductor and plasma boundary cross sections to the examples of Table I.

7. Possible Extensions

The computations may easily be generalized to a fully three-dimensional configuration. The integral equation (3.2) is just a special case of a three-dmiensional integral equation (see [14]), and an analogous representation to (5.1) and (5.2) may easily be given for a torus without rotational symmetry. Unfortunately, the integral equation then is no longer a scalar one.

Another possible extension could be the treatment of a diffuse boundary in the form of a set of nested sharp boundaries. In this case we have plasma boundaries S^i, $i = 1,\dots, n$, such that S^i is completely interior to S^{i+1}. The magnetic field B is given by $B = B^0 + B^1 + \cdots + B^n$, where $B^i = t^i/(2\pi d) \cdot e + \bar{B}^i$, analog to (2.11), (2.12), with $\bar{B}^0$ vanishing; and (2.14) is replaced by

$$\left(\sum_{k=0}^{i} B^k\right)^2 - \left(\sum_{k=0}^{i-1} B^k\right)^2 - 8\pi\, \Delta^i p = 0 \qquad \text{on } S^i,$$

where $\Delta^i p$ is the pressure jump accross the ith surface.

As all the boundaries S^i lie close to each other, the evaluation of their coefficients takes much less time than the determination of n totally different boundaries and may in some instances even be done by interpolation. The determination of the distribution of the pressure jumps and the azimuthal current on the different surfaces has to be done separately.

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